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Existence of the Mild Solutions for an Impulsive Fractional Differential Inclusions in Banach Space

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Abstract

Our main contribution in this paper is to prove the existence of the mild solutions for an impulsive fractional differential inclusions involving the Caputo derivative in Banach spaces. The results are obtained by using fractional calculation, operator semigroups and Leray Schauder's fixed point theorem. An example is given to illustrate the theory.

Key Words: Mild Solutions, Fractional Impulsive Differential Inclusions, Operator Semigroup

AMS Subject Classifications: 34A08, 34A12, 34K45, 34A60

1 Introduction

This paper is concerned with the existence of the mild solutions for an impulsive fractional differential inclusions of the form:

$$\begin{aligned} &{}^{c}D_t^q x(t) \in Ax(t) + F(t, x(t)) \\ &\Delta x(t_k) = I_k(x(t_k^-)); k = 1, 2, ..., m \qquad (1.1) \\ &x(0) = x_0;\ 0 < T < \infty \end{aligned}$$

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where $A : D(A) \subset X \to X$ is a closed, densely defined linear operator and infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on Banach space X. ${}^cD_t^q$ denotes the Caputo fractional derivative of order q, the state $x(.)$ takes values in Banach space X, $F : J \times X \to 2^X \backslash \{\phi\}$ is a nonempty, bounded, closed and convex multivalued map. Here, $0 = t_0 < t_1 < t_2 < .... < t_m < t_{m+1} = T$; $I_k \in C(X, X)$, $k = 1, 2, ..., m$ are bounded functions, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$; $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represents the right and left limits of $x(t)$ at $t = t_k$ respectively.

Differential equations with fractional order have recently proved valuable tools in the modeling of many physical phenomena ([17],[18],[33]). There has also been a significant theoretical development in fractional differential equations in recent years; see the monographs of Kilbas et al. ([1]), Miller and Ross([36]), Podlubny([19]), Samako et al. ([46]), and the papers of Bai and Lu ([59]), Diethelm et al. ([33],[34],[35]), El-Sayed ([8],[7],[6]), El-Sayed and Ibrahim ([9]), Kilbas and Trujillo ([3]), Podlubny et al. ([20]), and Yu and Gao ([14]).

During the last ten years, impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensely studied by many mathematicians. At present the foundations of the general theory are already laid, and many of them are investigated in detail in the book of Aubin ([24]), Benchohra et al. ([43]), and in the papers of Henderson and Ouahab ([22]), Graef et al. ([30],[31],[29]), and the references therein.

Differential equations with impulses were considered for the first time in the 1960's by Milman and Myshkis ([52],[51]). A period of active research, primarily in Eastern Europe from 1960-1970, culminated with the monograph by Halanay and Wexler ([4]).

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters.These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligiblein comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume that these perturbations act instantaneously or in the form of "impulses". As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention. Works recognized as landmark contributions include

([15],[57],[45],[11]). There are also many different studies in biology and medicine for which impulsive differential equations are good models. In the periodic treatment of some diseases, impulses correspond to administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs.

Very recently, some basic theory for initial-value problems for fractional differential equations and inclusions involving the Riemann-Liouville differential operator was discussed by Benchohra et al. ([41]), Lakshmikantham ([53]), and Lakshmikantham and Vastala ([54],[56],[55]). El-Sayed and Ibrahim ([9]) initiated the study of fractional multivalued differential inclusions. Recently, fractional functional differential equations and inclusions with standard Riemann-Liouville and Caputo derivatives with differences conditions were studied by Benchohra et al. ([39],[41],[42]), Henderson and Ouahab ([21]), and Ouahab ([12]). Bouzaroura and Mazouzi ([13]) has proved the improved existence results for impulsive fractional differential equations.

In [13] authors have claimed that most of the published papers dealing with impulsive differential equation of fractional orders are not mathematically correct and they have introduced a new class of impulsive fractional problems with several fractional orders.

In this paper, our main contribution is to prove the existence and uniqueness of mild solutions for an impulsive fractional differential inclusions involving the Caputo derivative with Sectorial operator by the new concept introduced by [13].

2 Preliminaries

In this section, we shall introduce some basic definitions, notations and lemmas from mutli-valued analysis which are used throughout this paper.
$C(J,X)$ denotes the Banach space of all continuous functions from J into X with the norm

$$\|x\|_\infty := \sup\Big\{\|x(t)\| : t \in J\Big\}.$$

In order to define the solution of (1.1) we shall consider the space of functions

$\Omega = \{x : J \to X :$ there exist $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$ such that $t_k = \tau_k(x(t_k)),\ x(t_k^+)$ exists, $k = 0, 1, 2, ..., m$ and $x_k \in C((t_k, t_{k+1}], X),\ k = 0, 1, 2, ..., m\}$,

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where x_k is the restriction of x over $(t_k, t_{k+1}],\ k = 0, ..., m.$

$L^1(J, X)$ denotes the Banach space of measurable functions $x : J \to X$which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int\limits_0^T |x(t)|dt$ for all $x \in L^1(J, X)$.

Let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_{cp} = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex(closed) for all $x \in X$, G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x \in B}\{sup\{\|x\| : x \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X, if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*, y_n \in G(x_n)$) imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by F in G. A multivalued map $G : [0, 1] \to P_{cl}(X)$ is said to be measurable if for every $y \in X$, the function $t \mapsto d(y, G(t)) = inf\{|y - z| : z \in G(t)\}$ is measurable.

For more details on multivalued map see the books of Aubin and Frankowska ([23]), Deimling ([32])and Hu and Papageorgiou ([50]).

DEFINITION 2.1 *A multivalued map $F : J \times X \to \mathcal{P}(X)$ is said to be L^1-Carathéodory if*
(i) $t \mapsto F(t, u)$ is measurable for each $u \in X$;
(ii) $u \mapsto F(t, u)$ is upper semicontinuous for almost all $t \in J$;
(iii) for each $q > 0$, there exists $\varphi \in L^1(J, X)$ such that
$\|F(t, u)\| = sup\{|v|\ v \in F(t, u)\} \leq \varphi(t)$ for all $|u| \leq q$ and for a.e. $t \in J$.

For each $x \in \Omega$, define the set of selections of F by

$$S_{F,x} = \{v \in L^1(J, X) : v(t) \in F(t, x(t)) a.e. t \in J\} \text{ is nonempty.}$$

4

LEMMA 2.2 *([5]) Let $F : J \times X \to P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let $\mathcal{L}$ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$, then the operator*

$$\mathcal{L} \circ S_F : C(J, X) \to P_{cp,c}(C(J, X)), x \mapsto (\mathcal{L} \circ S_F)(x) := \mathcal{L}(S_{F,x})$$

is a closed graph operator in $C(J, X \times C(J, X))$.

LEMMA 2.3 *([49]) Let $\alpha > 0$, then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1t + c_2t^2 + ... + c_{n-1}t^{n-1}$, $c_i \in X$, $i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

LEMMA 2.4 *([49]) Let $\alpha > 0$, then*

$$I^{\alpha c}D^\alpha h(t) = h(t) + c_0 + c_1t + c_2t^2 + ... + c_{n-1}t^{n-1}$$

for some $c_i \in X$, $i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

DEFINITION 2.5 *([2],[57]) The fractional (arbitrary) order integral of the function $h \in L^1([a, b], X)$ is defined by*

$$I_a^\alpha h(t) = \int_a^\alpha \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function.
When $a = 0$, we write $I^\alpha h(t) = [h \times \varphi_\alpha](t)$, where

$$\varphi_\alpha(t) = \tfrac{t^{\alpha-1}}{\Gamma(\alpha)} \text{ for } t > 0, \text{ and } \varphi_\alpha(t) = 0 \text{ for } t \le 0.$$

$$\varphi_\alpha(t) \to \delta(t) \text{ as } \alpha \to 1, \text{ where } \delta \text{ is the delta function.}$$

DEFINITION 2.6 *([2],[57]) For a function h given on the interval $[a, b]$, the αth Riemann-Liouville fractional-order derivative of h, is defined by*

$$(D_{a+}^\alpha h)(t) = \tfrac{1}{\Gamma(n-\alpha)} \left(\tfrac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

DEFINITION 2.7 *([2],[57]) For a function h given on the interval $[a, b]$, the αth Caputo fractional-order derivatives of h, is defined by*

$$({}^cD_{a+}^\alpha h)(t) = \tfrac{1}{\Gamma(n-\alpha)} \left(\tfrac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

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3 Existence of Solutions

Now, we define solution of problem (1.1)

DEFINITION 3.1 *A function $x \in \Omega$ is said to be a mild solution of system (1.1) if $x(0) = x_0$ and there exists $f \in L^1(J, X)$ such that $f(t) \in F(t, x(t))$ on $t \in J$ and*

$$
\begin{aligned}
x(t) &= x_0 + \frac{1}{\Gamma\alpha_0}\int_0^t (t-s)^{\alpha_0-1} S(t-s)[Ax(s) + f(s)]ds;\ t \in [a, t_1] \\
x(t) &= x_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma\alpha_k}\int_{t_k}^{t} (t-s)^{\alpha_k-1} S(t_k-s)[Ax(s)+v(s)]ds \\
&+ \sum_{i=1}^{k}\frac{1}{\Gamma\alpha_{i-1}}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha_{i-1}-1} S(t_i-s)[Ax(s)+v(s)]ds;\ t \in J_k;
\end{aligned}
$$

where $k = 1, 2, ..., m;\ v \in S_{F,x}$.

Let us consider the following hypotheses which are assumed to prove theorem

(H1): The function $F : J \times X \to P_{cp,c}(X)$ be an L^1-Carathéodory.

(H2): There exists a continuous non-decreasing function $\varphi : [0, \infty) \to (0, \infty)$ and a function $p \in C(J, X)$ such that

$$||F(t, x)|| \leq p(t)\varphi(|x|) \text{ for each } (t, x) \in J \times X$$

$$\rho^0 = sup\{p(t) : t \in J\},\ ||x||_\infty = sup\{|x(t)| :\ t \in J\} \leq q.$$

(H3): $A : D(A) \subset X \to X$ is a continuous bounded linear operator and there exists a constant $M_1 > 0$ such that $\|A\| < M_1$.

(H4): $f : J \times X \to 2^X \backslash \{\phi\}$ is continuous and there exists N_1 and M_2 such that $||f|| \leq N_1$ and $|S(t_i - s)| \leq M_2$.

THEOREM 3.2 *Assume that hypotheses $(H1)-(H4)$ hold, then (1.1) has atleast one solution, provided*

(a) *there exist constant l^* such that*

$$\left|\sum_{i=1}^{k} I_i(x(t_i^-))\right| \leq \sum_{i=1}^{k} l^*|x(t_i^-)| \leq \sum_{i=1}^{k} l^* q \leq k l^* q$$

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(b) *for any $q > 0$, there exists a positive constant r such that*

$$\begin{aligned} |z(t)| &\leq |x_0| + kl^*q + \left(\frac{(M_1 q + \rho^0\varphi(q))}{\Gamma\alpha_k}\right)T^{\alpha_k} \\ &+ \sum_{i=1}^{k}\left(\frac{(M_1 q + \rho^0\varphi(q))}{\Gamma\alpha_{i-1}}\right)T^{\alpha_{i-1}} \\ &\leq r \end{aligned}$$

(c) *there exists a number $\bar{M} > 0$, such that*

$$\frac{\bar{M}}{|x_0| + kl^*q + M_2(M_1 q + \rho^0\varphi(q))\left(\frac{T^{\alpha}}{\Gamma\alpha_k} + \sum_{i=1}^{k}\frac{T^{\alpha_{i-1}}}{\Gamma\alpha_{i-1}}\right)} > 1$$

Proof: Consider the operator $N : C(J, X) \to \mathcal{P}(C(J, X))$ defined by

$$\begin{aligned} N(x) = \Big\{z &\in C(J,X) : z(t) = x_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma\alpha_k}\int_{t_k}^{t}(t-s)^{\alpha_k - 1}S(t_k - s)[Ax(s) \\ &+ v(s)]ds + \sum_{i=1}^{k}\frac{1}{\Gamma\alpha_{i-1}}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha_{i-1}-1}S(t_i - s)[Ax(s) + v(s)]ds\Big\}, \end{aligned}$$

we shall show that N satisfies the assumptions of nonlinear alternative of Leray-Schauder type.

Step 1: $N(x)$ is convex for each $x \in C(J, X)$ if $z_i \in N(x)$ then there exists $A_i, v_i;\ i = 1, 2$ such that

$$\begin{aligned} z_1(t) &= x_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma\alpha_k}\int_{t_k}^{t}(t-s)^{\alpha_k-1}S(t_k - s)[A_1x(s) + v_1(s)]ds \\ &+ \sum_{i=1}^{k}\frac{1}{\Gamma\alpha_{i-1}}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha_{i-1}-1}S(t_i - s)[A_1x(s) + v_1(s)]ds \end{aligned}$$

and

$$\begin{aligned} z_2(t) &= x_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma\alpha_k}\int_{t_k}^{t}(t-s)^{\alpha_k-1}S(t_k - s)[A_2x(s) + v_2(s)]ds \\ &+ \sum_{i=1}^{k}\frac{1}{\Gamma\alpha_{i-1}}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha_{i-1}-1}S(t_i - s)[A_2x(s) + v_2(s)]ds \end{aligned}$$

Let $0 \leq \lambda \leq 1$. Then for each $t \in J$
We have,

$$\begin{aligned}
&\lambda z_1(t) + (1-\lambda) z_2(t) \\
&= \Bigg\{ x_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) + \tfrac{1}{\Gamma\alpha_k} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} S(t_k - s)[\lambda A_1 x(s) + \lambda v_1(s)] ds \\
&+ \sum_{i=1}^{k} \tfrac{1}{\Gamma\alpha_{i-1}} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} S(t_i - s)[\lambda A_1 x(s) + \lambda v_1(s)] ds \Bigg\} \\
&+ \Bigg\{ x_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) + \tfrac{1}{\Gamma\alpha_k} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} S(t_k - s)[(1-\lambda) A_2 x(s) + (1-\lambda) v_2(s)] ds \\
&+ \sum_{i=1}^{k} \tfrac{1}{\Gamma\alpha_{i-1}} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} S(t_i - s)[(1-\lambda) A_2 x(s) + (1-\lambda) v_2(s)] ds \Bigg\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda z_1(t) + (1-\lambda) z_2(t) &= x_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma\alpha_k} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} S(t_k - s) \\
&\quad [\{\lambda A_1 + (1-\lambda) A_2\} x(s) + \{\lambda v_1 + (1-\lambda) v_2\}(s)] ds
\end{aligned}$$

Since $S_{F,x}$ is convex(because F has convex value) then $\lambda z_1 + (1-\lambda) z_2 \in N(x)$.

Step 2: N maps bounded sets into bounded sets in $C(J, X)$.
To show that for any $q > 0$, there exists a positive constant M such that

$$x \in B_q = \{x \in C(J, X) : \|x\|_\infty \leq q\}$$

We have, $\|N(x)\|_\infty \leq M$ then for each $z \in N(x)$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned}
z(t) &= x_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma\alpha_k} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} S(t_k - s)[Ax(s) + v(s)] ds \\
&+ \sum_{i=1}^{k} \frac{1}{\Gamma\alpha_{i-1}} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} S(t_i - s)[Ax(s) + v(s)] ds
\end{aligned}$$

$$\begin{aligned}
|z(t)| &\leq |x_0| + \left| \sum_{i=1}^{k} I_i(x(t_i^-)) \right| + \left| \frac{1}{\Gamma\alpha_k} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} S(t_k - s)[Ax(s) + v(s)] ds \right| \\
&+ \left| \sum_{i=1}^{k} \frac{1}{\Gamma\alpha_{i-1}} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} S(t_i - s)[Ax(s) + v(s)] ds \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \ |x_0| + kl^*q + \frac{1}{\Gamma\alpha_k}\int_{t_k}^{t} |(t-s)^{\alpha_k - 1}||S(t_k - s)|[|Ax(s)| + |v(s)|]ds \\
&+ \ \sum_{i=1}^{k} \frac{1}{\Gamma\alpha_{i-1}}\int_{t_{i-1}}^{t_i} |(t_i-s)^{\alpha_{i-1} - 1}||S(t_i - s)|[|Ax(s) + v(s)|]ds \\
&\leq \ |x_0| + kl^*q + \frac{M_1 q + \rho^0\varphi(s)}{\Gamma\alpha_k}\int_{t_k}^{t} |(t-s)^{\alpha_k - 1}||S(t_k - s)|ds \\
&+ \ \sum_{i=1}^{k} \frac{M_1 q + \rho^0\varphi(s)}{\Gamma\alpha_{i-1}}\int_{t_{i-1}}^{t_i} |(t_i-s)^{\alpha_{i-1} - 1}||S(t_i - s)|ds \\
&\leq \ |x_0| + kl^*q + \left(\frac{M_1 q + \rho^0\varphi(s)}{\Gamma\alpha_k}\right)M_2 T^{\alpha_k} + \sum_{i=1}^{k}\left(\frac{M_1 q + \rho^0\varphi(s)}{\Gamma\alpha_{i-1}}\right)M_2 T^{\alpha_{i-1}} \\
&\leq \ r
\end{aligned}
$$

Step 3: N maps bounded sets into equicontinuous sets of $C(J, X)$.

$$
\begin{aligned}
z(t) \ &= \ x_0 + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma\alpha_k}\int_{t_k}^{t}(t-s)^{\alpha_k - 1}S(t_k - s)[Ax(s) + v(s)]ds \\
&+ \ \sum_{i=1}^{k}\frac{1}{\Gamma\alpha_{i-1}}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha_{i-1}-1}S(t_i - s)[Ax(s) + v(s)]ds; \text{for } t_1 < t_2
\end{aligned}
$$

$$
\begin{aligned}
&|z(t_2) - z(t_1)| \\
&\leq \tfrac{1}{\Gamma\alpha_k}\int_{t_k}^{t_1} |(t_2 - s)^{\alpha_k - 1} - (t_1 - s)^{\alpha_k - 1}||S(t_k - s)||[Ax(s) + v(s)]|ds \\
&\qquad + \tfrac{1}{\Gamma\alpha_k}\int_{t_1}^{t_2} |(t_2 - s)^{\alpha_k - 1}||S(t_k - s)||[Ax(s) + v(s)]|ds \\
&\leq \left(\tfrac{M_1 q + \rho^0\varphi(q)}{\Gamma\alpha_k}\right)M_2\int_{t_k}^{t_1} |(t_2 - s)^{\alpha_k - 1} - (t_1 - s)^{\alpha_k - 1}|ds \\
&\qquad + \left(\tfrac{M_1 q + \rho^0\varphi(q)}{\Gamma\alpha_k}\right)M_2\int_{t_1}^{t_2} |(t_2 - s)^{\alpha_k - 1} - (t_1 - s)^{\alpha_k - 1}|ds
\end{aligned}
$$

As $t_1 \to t_2$ then $|z(t_2) - z(t_1)| \to 0$.
By Steps 1 to 3 and from Arzela-Ascoli theorem, we can conclude that $N : C(J, X) \to \mathcal{P}(C(J, X))$ is completely continuous.

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Step 4: N has a closed graph.
Let $x_n \to x_*$ and $z_n \to z_*$ then for $z_n \in N(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in J$
To prove:- $z_* \in N(x_*)$

$$\begin{aligned} z_n(t) &= x_0 + \sum_{i=1}^{k} I_i(x_n(t_i^-)) + \frac{1}{\Gamma\alpha_k}\int_{t_k}^{t}(t-s)^{\alpha_k - 1}S(t_k - s)[Ax_n(s) + v_n(s)]ds \\ &+ \sum_{i=1}^{k}\frac{1}{\Gamma\alpha_{i-1}}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha_{i-1}-1}S(t_i - s)[Ax_n(s) + v_n(s)]ds \end{aligned}$$

$$\begin{aligned} z_*(t) &= x_0 + \sum_{i=1}^{k} I_i(x_*(t_i^-)) + \frac{1}{\Gamma\alpha_k}\int_{t_k}^{t}(t-s)^{\alpha_k - 1}S(t_k - s)[Ax_*(s) + v_*(s)]ds \\ &+ \sum_{i=1}^{k}\frac{1}{\Gamma\alpha_{i-1}}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha_{i-1}-1}S(t_i - s)[Ax_*(s) + v_*(s)]ds \end{aligned}$$

$$\begin{aligned} ||z_n(t) - z_*(t)|| &= \Bigg\| \frac{1}{\Gamma\alpha_k}\int_{t_k}^{t}(t-s)^{\alpha_k - 1}S(t_k - s)[A(x_n - x_*)(s) + (v_n - v_*)(s)]ds \\ &+ \sum_{i=1}^{k}\frac{1}{\Gamma\alpha_{i-1}}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha_{i-1}-1}S(t_i - s)[A(x_n - x_*)(s) + (v_n - v_*)(s)]ds \\ &+ \sum_{i=1}^{k} I_i(x_n(t_i^-) - x_*(t_i^-))\Bigg\| \\ &\leq \frac{M_2 T^{\alpha}}{\Gamma\alpha_k}\|[A(x_n - x_*)(s) + (v_n - v_*)(s)]\| \\ &+ \sum_{i=1}^{k}\frac{M_2 T^{\alpha_{i-1}}}{\Gamma\alpha_{i-1}}\|[A(x_n - x_*)(s) + (v_n - v_*)(s)]\| + kl^* q \\ &\leq \frac{M_2 T^{\alpha}}{\Gamma\alpha_k}\|A\|\|(x_n(s) - x_*(s))\| + \|(v_n(s) - v_*(s))\| \\ &+ \sum_{i=1}^{k}\frac{M_2 T^{\alpha_{i-1}}}{\Gamma\alpha_{i-1}}\|A\|\|(x_n(s) - x_*)(s)\| + \|(v_n - v_*(s))\| + kl^* q \end{aligned}$$

$||z_n(t) - z_*(t)|| \to 0$ as $n \to \infty$, as $x_n \to x_*$ and $v_n \to v_*$.

Consider, the continuous linear operator $\Gamma : L^1(J, X) \to C(J, X)$ defined by

$$\begin{aligned} v \to (\Gamma v)(t) &= \int_{t_k}^{t} (t-s)^{\alpha_k - 1} S(t_k - s)[Ax(s) + v(s)]ds \\ &+ \sum_{i=1}^{k} \frac{1}{\Gamma\alpha_{i-1}} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} S(t_i - s)[Ax(s) + v(s)]ds \end{aligned}$$

from lemma 2.2, Γ is a closed graph operator in $C(J, X \times C(J, X))$.

Step 5: A priori bounds on solutions.
Let $x \in C(J, X)$ be such that $x \in \lambda N(x)$ for some $\lambda \in (0, 1)$ then there exists $v \in L^1(J, X)$ with $v \in S_{F,x}$ such that for each $t \in J$

$$\begin{aligned} x(t) &= \lambda x_0 + \sum_{i=1}^{k} I_i(\lambda x(t_i^-)) + \frac{1}{\Gamma\alpha_k} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} S(t_k - s)[A\lambda x(s) + \lambda v_(s)]ds \\ &+ \sum_{i=1}^{k} \frac{1}{\Gamma\alpha_{i-1}} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} S(t_i - s)[A\lambda x_(s) + \lambda v_(s)]ds \\ |x(t)| &\leq |x_0| + \sum_{i=1}^{k} |I_i(x(t_i^-))| + \frac{1}{\Gamma\alpha_k} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} S(t_k - s)|[Ax(s) + v_(s)]|ds \\ &+ \sum_{i=1}^{k} \frac{1}{\Gamma\alpha_{i-1}} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} S(t_i - s)|[Ax_(s) + v_(s)]|ds \\ &\leq |x_0| + kl^*q + \frac{M_2 T^{\alpha_k}}{\Gamma\alpha_k}|[Ax(s) + v_(s)]| + \sum_{i=1}^{k} \frac{M_2 T^{\alpha_{i-1}}}{\Gamma\alpha_{i-1}}|[Ax_(s) + v_(s)]| \\ \|x\|_\infty &\leq |x_0| + kl^*q + \frac{M_2 T^{\alpha_k}}{\Gamma\alpha_k}(M_1 q + \rho^0\varphi(q)) + \sum_{i=1}^{k} \frac{M_2 T^{\alpha_{i-1}}}{\Gamma\alpha_{i-1}}(M_1 q + \rho^0\varphi(q)) \end{aligned}$$

$$\frac{\|x\|_\infty}{|x_0| + kl^*q + M_2(M_1 q + \rho^0\varphi(q))\left(\frac{T^\alpha}{\Gamma\alpha_k} + \sum\limits_{i=1}^{k} \frac{T^{\alpha_{i-1}}}{\Gamma\alpha_{i-1}}\right)} \leq 1$$

Let $U = \{x \in PC(J, X) : ||x||_\infty < \bar{M}\}$
The operator $F : \bar{U} \to PC(J, X)$ is continuous and completely continuous from the choice of U, there is no $x \in \partial U$ such that $x = \lambda F(x)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type. We denote that F has a fixed point x in $\bar{U}$ which is a solution of the problem.

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4 Example:

Consider, the second fractional impulsive problem

$$\begin{cases} {}^cD_t^q u(t) = \frac{|u(t)|}{(1+e^t)(1+|u(t)|)}, \ q \in (0,1), \ t \in [0,T]\backslash\{t\}, \\ u(0) = 0, \\ u(t_1^+) = u(t_1^-) + \frac{1}{2}. \end{cases}$$

Set

$$f_1(t,u) = \frac{u}{(1+e^t)(1+u)}, \ (t,u) \in [0,T] \times [0,\infty).$$

Obviously, for all $u \in [0,\infty)$ and each $t \in [0,T]$,

$$|f_1(t,u)| = \frac{1}{(1+e^t)}\left|\frac{u}{1+u}\right| \leq \frac{1}{4}|u|.$$

Thus, all the assumptions in theorem (3.2) are satisfied, our result can be applied to the problem (1.1).

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